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Recursive calculation of the R-matrices of q-deformed algebras

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Abstract. We show that the pentagonal equation, relating R-matrices and vector coupling coefficients, can be used to calculate recursively general R-matrices from those containing the primitive irrep. All R-matrices for $\mathrm{su}(2)_q$ are calculated by this method. As an illustration of the more general case the primitive vector coupling coefficients for $\mathrm{su}(3)_q$ are obtained and used to derive formulae for the R-matrices with a certain restriction on the representations.

1. Introduction

Yang–Baxter equations arise in statistical mechanics (Baxter 1982), conformal field theory (Alvarez-Gaume *et al* 1990a, de Vega 1989, Witten 1990) and in connection with braid groups (Akutsu and Wadati 1987). The search for solutions, the R-matrices, to quantum Yang–Baxter equations inspired the development of quantum groups (Sklyanin 1982, Kulish and Reshetikhin 1981). 'Quantum groups' are one-parameter deformations of universal enveloping algebras of Lie algebras. Solutions to the Yang–Baxter equation without spectral parameter may be based on vector coupling coefficients of these q-deformed algebras. Representations of the braid group may then be obtained from R-matrices.

The operator form of universal R-matrices for all quantum groups associated with finite-dimensional Lie algebras are known (Burroughs 1990, Jimbo 1985, 1987, Rosso 1989). A more explicit form is known for the fundamental representations of $\mathrm{su}(n)_q$ and other q-deformed Lie groups (Reshetikhin 1988). The explicit form of all the R-matrices for $\mathrm{su}(2)_q$ has been calculated (Nomura 1989). Some R-matrices for representations other than the fundamental have been found for $\mathrm{su}(3)_q$ (Ma 1990a, b). These matrices have been obtained by calculating the appropriate vector coupling coefficients of the quantum group and then summing over products of all the coefficients for particular irreps. Only three representations were considered in Ma (1990a, b) because no general form is known for the vector coupling coefficients of $\mathrm{su}(3)_q$.

In this paper we introduce a new recursive method for calculating R-matrices which requires only a few vector coupling coefficients to be known. The primitive coupling coefficients for $\mathrm{su}(2)_q$ have been obtained in Lienert and Butler (1992) while all the coefficients required for the $\mathrm{su}(3)_q$ R-matrix calculation are given in section 4. The recursive method is illustrated by calculating R-matrices for $\mathrm{su}(2)_q$ and a class of R-matrices for $\mathrm{su}(3)_q$.

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2. Properties of R-matrices

The q-deformed algebras are quasi-triangular Hopf algebras (Abe 1980). An element, $\mathcal{R}$, called the universal R-matrix, can be constructed for such algebras (Drinfeld 1985). Restrictions of the universal R-matrix to irreps may be defined from $\mathcal{R}$ by

$$R^{\lambda_1\lambda_2} = P^{\lambda_1\lambda_2}\mathcal{R} \tag{1}$$

where P is the permutation operator on the tensor product of the irrep spaces. These matrices satisfy the Yang–Baxter equation without the spectral parameter (Reshetikhin 1988, Hou *et al* 1990), namely

$$R^{\lambda_2\lambda_1}R^{\lambda_3\lambda_1}R^{\lambda_3\lambda_2} = R^{\lambda_3\lambda_2}R^{\lambda_3\lambda_1}R^{\lambda_2\lambda_1}\,. \tag{2}$$

The R-matrices effect a $q \to 1/q$ transformation in the vector coupling coefficients (Reshetikhin 1988),

$$\left(R_q^{\lambda_1\,\lambda_2}\right)^{m_1\;m_2}_{m_1'\;m_2'}\ {}_q\langle\lambda_1 m_1\lambda_2 m_2|r\lambda m\rangle = q^{c(\lambda_1)+c(\lambda_2)-c(\lambda)/2}\ {}_{\frac{1}{q}}\langle\lambda_1 m_1'\lambda_2 m_2'|r\lambda m\rangle \tag{3}$$

where $c(\lambda)$ is the quadratic Casimir operator on λ. On using the orthogonality of the vector coupling coefficients, we have

$$\left(R_q^{\lambda_1\,\lambda_2}\right)^{m_1\;m_2}_{m_1'\;m_2'} = \sum_{r\lambda m} q^{c(\lambda_1)+c(\lambda_2)-c(\lambda)/2}\ {}_q\langle r\lambda m|\lambda_1 m_1\lambda_2 m_2\rangle\ {}_{\frac{1}{q}}\langle\lambda_1 m_1'\lambda_2 m_2'|r\lambda m\rangle\,. \tag{4}$$

The symmetries of the vector coupling coefficients (Lienert and Butler 1992) imply the following symmetries for the R-matrices:

$$\left(R_q^{\lambda_1\,\lambda_2}\right)^{m_1\;m_2}_{m_1'\;m_2'} = \left(R_q^{\lambda_2\,\lambda_1}\right)^{m_2'\;m_1'\;*}_{m_2\;m_1} \tag{5}$$

$$= \sum_{n_1'n_2'n_1n_2}\left(R_q^{\lambda_1^*\,\lambda_2^*}\right)^{n_1'\;n_2'}_{n_1\;n_2}{}_q(\lambda_1)^{m_1n_1}\,{}_{\frac{1}{q}}(\lambda_1)^{m_1'n_1'}\,{}_q(\lambda_2)^{m_2n_2}\,{}_{\frac{1}{q}}(\lambda_2)^{m_2'n_2'}\,; \tag{6}$$

where λ_1^* (or λ_2^*) is the highest weight of the representation dual to λ_1 (or λ_2).

The R-matrices and vector coupling coefficients satisfy the pentagonal relation (Reshetikhin 1988, Nomura 1989, Hou *et al* 1990)

$$\sum_{m_1,m_2,m'}\left(R_q^{\lambda_1\,\lambda}\right)^{m_1\;m'}_{m_1'\;m''}\left(R_q^{\lambda_2\,\lambda}\right)^{m_2\;m}_{m_2'\;m'}\ {}_q\langle\lambda_1 m_1\lambda_2 m_2|r\lambda_3 m_3\rangle = \sum_{m_3'}\ {}_q\langle\lambda_1 m_1'\lambda_2 m_2'|r\lambda_3 m_3'\rangle\left(R_q^{\lambda_3\,\lambda}\right)^{m_3\;m}_{m_3'\;m''}\,. \tag{7}$$

This equation is used in sections 3 and 5 as a recursion relation for finding general R-matrices.

3. Calculation of $su(2)_q$ R-matrices

The R-matrices containing the trivial irrep and the primitive irrep are calculated directly from (4), using the values of the trivial and primitive vector coupling coefficients (Lienert and Butler 1992).

The trivial R-matrix is 1, and the non-zero primitive R-matrices are easily computed. We have

$$\begin{aligned}\left(R_q^{\frac{1}{2}j_2}\right)^{\frac{1}{2}\ m}_{\frac{1}{2}\ m} &= \left(R_q^{\frac{1}{2}j_2}\right)^{-\frac{1}{2}\ -m}_{-\frac{1}{2}\ -m} \\ &= q^{\{c(\frac{1}{2})+c(j_2)-c(j_2+\frac{1}{2})\}}\ {}_q\langle\tfrac{1}{2}\tfrac{1}{2}\,j_2 m|j_2+\tfrac{1}{2}\,m+\tfrac{1}{2}\rangle\ {}_{\frac{1}{q}}\langle\tfrac{1}{2}\tfrac{1}{2}\,j_2 m|j_2+\tfrac{1}{2}\,m+\tfrac{1}{2}\rangle \\ &\quad + q^{\{c(\frac{1}{2})+c(j_2)-c(j_2-\frac{1}{2})\}}\ {}_q\langle\tfrac{1}{2}\tfrac{1}{2}\,j_2 m|j_2-\tfrac{1}{2}\,m+\tfrac{1}{2}\rangle\ {}_{\frac{1}{q}}\langle\tfrac{1}{2}\tfrac{1}{2}\,j_2 m|j_2-\tfrac{1}{2}\,m+\tfrac{1}{2}\rangle \\ &= \frac{1}{[2j_2+1]}\left\{q^{-j_2}[j_2+m+1]+q^{j_2+1}[j_2-m]\right\} = q^{-m} \end{aligned} \tag{8}$$

while a similar calculation gives

$$\left(R_q^{\frac{1}{2}j_2}\right)^{-\frac{1}{2}\ \ m}_{\frac{1}{2}\ \ m-1} = (q^{-1}-q)\left\{[j_2-m+1][j_2+m]\right\}^{\frac{1}{2}} q^{\frac{1}{2}} \tag{9}$$

where the q-numbers $[x]$ are defined by $[x] \equiv (q^x - q^{-x})/(q-q^{-1})$ and for $su(2)_q$, $c(j) = j(j+1)$.

A recursion relation for the general case can now be obtained by letting λ_1 be the primitive irrep $\frac{1}{2}$, and m_1' be $\frac{1}{2}$ in the pentagonal equation (7). Substituting for the primitive R-matrices and vector coupling coefficients, and changing notation (namely $j_1-\frac{1}{2}$ for λ_2, j_1 for λ_3, j_2 for λ, etc) gives the recursion relation

$$\begin{aligned}\left(R_q^{j_1 j_2}\right)^{n-i\ \ m}_{n\ \ m-i} q^{-n+j_1/2}[j_1+n]^{\frac{1}{2}} &= q^{-n-j_1+i+1/2}(q^{-1}-q) \\ &\quad \times\left\{[j_2-m+i][j_2+m-i+1][j_1-n+i]\right\}^{\frac{1}{2}}\left(R_q^{j_1-\frac{1}{2}j_2}\right)^{n-i+\frac{1}{2}\ \ m}_{n-\frac{1}{2}\ \ m-i+1} \\ &\quad + q^{-n+j_1-2m+3i/2}[j_1+n-i]^{\frac{1}{2}}\left(R_q^{j_1-\frac{1}{2}j_2}\right)^{n-i-\frac{1}{2}\ \ m}_{n-\frac{1}{2}\ \ m-i}. \end{aligned} \tag{10}$$

Iterating this expression $2j_1$ times, with the maximum value of $n = j_1$, gives R-matrices on the right-hand side of the form

$$\left(R_q^{0 j_2}\right)^{2j_1-i-t\ \ \ m}_{0\ \ \ m-i+2j_1-t}.$$

The only such R-matrix which is non-zero is the trivial R-matrix for which $2j_1-i-t = 0$. Substituting, we have

$$\left(R_q^{j_1 j_2}\right)^{j_1-i\ \ m}_{j_1\ \ m-i} = \left\{\frac{[2j_1]![j_2-m+i]![j_2+m]!}{[i]![2j_1-i]![j_2+m-i]![j_2-m]!}\right\}^{\frac{1}{2}} q^{\{-(2j_1-i)(2m-i)+i\}/2}. \tag{11}$$

To obtain the R-matrices for $n < j_1$, we rearrange the recursion relation (10), and substitute j_1 for $j_1 + \frac{1}{2}$ and i for $i+1$, to give

$$\begin{aligned}\left(R_q^{j_1 j_2}\right)^{n-i\ \ m}_{n\ \ m-i} &= \left\{\frac{[j_2+m-i-1]}{[j_1-n+i+1][j_2-m+i+1]}\right\}^{\frac{1}{2}}(q^{-1}-q)\\ &\quad\times\left\{[j_1+n+1]^{\frac{1}{2}}q^{-i-1+2j_1/2}\left(R_q^{j_1+\frac{1}{2}\,j_2}\right)^{n-i-\frac{1}{2}\ \ m}_{n+\frac{1}{2}\ \ m-i-1}\right.\\ &\quad\left.-[j_1+n-i]^{\frac{1}{2}}(q^{-1}-q)q^{-2m+2i+2j_1+2/2}\left(R_q^{j_1 j_2}\right)^{n-i-1\ \ m}_{n\ \ m-i-1}\right\}.\end{aligned} \tag{12}$$

On iteration $m-i+j_2$ times we obtain an expression for $\left(R_q^{j_1 j_2}\right)^{n-i\ \ m}_{n\ \ m-i}$ in terms of a sum of R-matrices of the form

$$\left(R_q^{j_1+(m-i+j_2-t)/2\,j_2}\right)^{n+(m-i+j_2-t)/2\ \ m}_{n-(m+i+j_2+t)/2\ \ -j_2}.$$

The symmetries of the R-matrix, (5) and (6), and the result (11) allows us to evaluate the right-hand side. The summation can be performed by using the q-equivalent of the binomial coefficient sum rule (Andrews 1976). The resulting algebraic expression may be simplified to give the general R-matrices for $\mathrm{su}(2)_q$ as

$$\begin{aligned}\left(R_q^{j_1 j_2}\right)^{n-i\ \ m}_{n\ \ m-i} &= (q^{-1}-q)^i q^{\{-(2m-i)(2n-i)+i\}/2}\\ &\quad\times\frac{1}{[i]!}\left\{\frac{[j_1+n]![j_1-n+i]![j_2-m+i]![j_2+m]!}{[j_1-n]![j_1+n-i]![j_2+m-i]![j_2-m]!}\right\}^{\frac{1}{2}}\end{aligned} \tag{13}$$

which agrees with the result obtained by Nomura (1989), when differences in the definition of q are taken into account.

4. Primitive coupling coefficients for $\mathrm{su}(3)_q$

The $\mathrm{su}(3)_q$ representations are labelled by their Young tableaus, $\lambda = (h_1, h_2)$. They can be obtained in a basis $\mathrm{u}(1)_q \times \mathrm{su}(2)_q$ as $|\lambda\sigma t\tau\rangle$, where σ is a $\mathrm{u}(1)_q$ representation corresponding to hypercharge in $\mathrm{su}(3)$, t is the $\mathrm{su}(2)_q$ label of isospin and τ its z-component. This choice of basis is the same as that of Ma (1990a,b). The primitive irrepresentation, ϵ, is chosen to be $(1,0)$, its conjugate being $(1,1)$.

From Reshetikhin (1988) the trivial vector-coupling coefficient is

$${}_q\langle\lambda\sigma t\tau;\lambda^*\!-\!\sigma t\!-\!\tau|0000\rangle = \phi\frac{q^{3\sigma-2\tau/2}}{|\lambda|_q^{\frac{1}{2}}} \tag{14}$$

where the q-dimension of λ is $|\lambda|_q = [h_1-h_2+1][h_1+2][h_2+1]/[2]$ (Alvarez-Gaume *et al* 1990a, b) and ϕ is a phase.

The primitive vector-coupling coefficients for $\mathrm{su}(3)_q$ are calculated from the orthogonality and symmetry properties of the coefficients following the method of

Lienert and Butler (1992). Phases are chosen to agree with those in su(3) (Moshinsky 1962). With the $\mathrm{su}(2)_q$ primitive coefficients factored out, we have

$$
{}_q\langle (h_1,h_2)\,\sigma{+}\tfrac{2}{3}\,t;(1,0)\,{-}\tfrac{2}{3}\,0|(h_1+1,h_2)\,\sigma\,t\rangle
= q^{\frac{1}{3}h_1-\frac{1}{6}h_2+\frac{1}{2}\sigma+\frac{1}{3}}
\times\left\{\frac{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+\frac{2}{3}][\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma+t+\frac{5}{3}]}{[h_1-h_2+1][h_1+2]}\right\}^{\frac{1}{2}} \tag{15}
$$

$$
{}_q\langle (h_1,h_2)\,\sigma{+}\tfrac{2}{3}\,t;(1,0)\,{-}\tfrac{2}{3}\,0|(h_1,h_2+1)\,\sigma\,t\rangle
= q^{-\frac{1}{6}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-\frac{1}{6}}
\times\left\{\frac{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t+\frac{1}{3}][-\frac{1}{3}h_1+\frac{2}{3}h_2-\frac{1}{2}\sigma+t+\frac{2}{3}]}{[h_1-h_2+1][h_2+1]}\right\}^{\frac{1}{2}} \tag{16}
$$

$$
{}_q\langle (h_1,h_2)\,\sigma{+}\tfrac{2}{3}\,t;(1,0)\,{-}\tfrac{2}{3}\,0|(h_1-1,h_2-1)\,\sigma\,t\rangle
= -\,q^{-\frac{1}{6}h_1-\frac{1}{6}h_2+\frac{1}{2}\sigma-\frac{2}{3}}
\times\left\{\frac{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-t+\frac{1}{3}][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+\frac{4}{3}]}{[h_1+2][h_2+1]}\right\}^{\frac{1}{2}} \tag{17}
$$

$$
{}_q\langle (h_1,h_2)\,\sigma{-}\tfrac{1}{3}\,t{-}\tfrac{1}{2};(1,0)\,\tfrac{1}{3}\,\tfrac{1}{2}|(h_1+1,h_2)\,\sigma\,t\rangle
= q^{-\frac{1}{3}h_1+\frac{1}{6}h_2+\frac{1}{4}\sigma+\frac{1}{2}t-\frac{1}{3}}\Bigg\{[\tfrac{1}{3}h_1-\tfrac{2}{3}h_2+\tfrac{1}{2}\sigma+t+\tfrac{1}{3}]
\times\frac{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma+t+\frac{5}{3}][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+\frac{4}{3}]}{[2t+1][h_1-h_2+1][h_1+2]}\Bigg\}^{\frac{1}{2}} \tag{18}
$$

$$
{}_q\langle (h_1,h_2)\,\sigma{-}\tfrac{1}{3}\,t{-}\tfrac{1}{2};(1,0)\,\tfrac{1}{3}\,\tfrac{1}{2}|(h_1,h_2+1)\,\sigma\,t\rangle
= -\,q^{\frac{1}{6}h_1-\frac{1}{3}h_2+\frac{1}{4}\sigma+\frac{1}{2}t+\frac{1}{6}}\Bigg\{[-\tfrac{1}{3}h_1+\tfrac{2}{3}h_2-\tfrac{1}{2}\sigma+t+\tfrac{2}{3}]
\times\frac{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+\frac{2}{3}][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+\frac{4}{3}]}{[2t+1][h_1-h_2+1][h_2+1]}\Bigg\}^{\frac{1}{2}} \tag{19}
$$

$$
{}_q\langle (h_1,h_2)\,\sigma{-}\tfrac{1}{3}\,t{-}\tfrac{1}{2};(1,0)\,\tfrac{1}{3}\,\tfrac{1}{2}|(h_1-1,h_2-1)\,\sigma\,t\rangle
= q^{\frac{1}{6}h_1+\frac{1}{6}h_2+\frac{1}{4}\sigma+\frac{1}{2}t+\frac{2}{3}}\Bigg\{[\tfrac{1}{3}h_1-\tfrac{2}{3}h_2+\tfrac{1}{2}\sigma+t+\tfrac{1}{3}]
\times\frac{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+\frac{2}{3}][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-t+\frac{1}{3}]}{[2t+1][h_1+2][h_2+1]}\Bigg\}^{\frac{1}{2}} \tag{20}
$$

$$
{}_q\langle (h_1,h_2)\,\sigma{-}\tfrac{1}{3}\,t{+}\tfrac{1}{2};(1,0)\,\tfrac{1}{3}\,\tfrac{1}{2}|(h_1+1,h_2)\sigma\,t\rangle
= q^{-\frac{1}{3}h_1+\frac{1}{6}h_2+\frac{1}{4}\sigma-\frac{1}{2}t-\frac{5}{6}}\Bigg\{[-\tfrac{1}{3}h_1+\tfrac{2}{3}h_2-\tfrac{1}{2}\sigma+t+\tfrac{2}{3}]
\times\frac{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+\frac{2}{3}][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-t+\frac{1}{3}]}{[2t+1][h_1-h_2+1][h_1+2]}\Bigg\}^{\frac{1}{2}} \tag{21}
$$

$$
{}_q\langle (h_1,h_2)\,\sigma{-}\tfrac{1}{3}\,t{+}\tfrac{1}{2};(1,0)\,\tfrac{1}{3}\,\tfrac{1}{2}|(h_1,h_2+1)\sigma t\rangle
$$

$$
= q^{\frac{1}{6}h_1-\frac{1}{3}h_2+\frac{1}{4}\sigma-\frac{1}{2}t-\frac{1}{3}}\left\{[\tfrac{1}{3}h_1-\tfrac{2}{3}h_2+\tfrac{1}{2}\sigma+t+\tfrac{1}{3}]\right.
$$

$$
\left.\times\frac{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma+t+\frac{5}{3}][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma-t+\frac{1}{3}]}{[2t+1][h_1-h_2+1][h_2+1]}\right\}^{\frac{1}{2}} \tag{22}
$$

$$
{}_q\langle (h_1,h_2)\,\sigma{-}\tfrac{1}{3}\,t{+}\tfrac{1}{2};(1,0)\,\tfrac{1}{3}\,\tfrac{1}{2}|(h_1-1,h_2-1)\sigma t\rangle
$$

$$
= q^{\frac{1}{6}h_1+\frac{1}{6}h_2+\frac{1}{4}\sigma-\frac{1}{2}t+\frac{1}{6}}\left\{[-\tfrac{1}{3}h_1+\tfrac{2}{3}h_2-\tfrac{1}{2}\sigma+t+\tfrac{2}{3}]\right.
$$

$$
\left.\times\frac{[\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma+t+\frac{5}{3}][\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+\frac{4}{3}]}{[2t+1][h_1+2][h_2+1]}\right\}^{\frac{1}{2}}. \tag{23}
$$

In the $q \to 1$ limit, the su(3) coefficients of Moshinsky (1962) are recovered.

5. Calculation of $su(3)_q$ R-matrices

As in the $su(2)_q$ case, the primitive R-matrices are calculated immediately from the primitive vector coupling coefficients. The non-zero primitive R-matrices are given below:

$$
\left(R_q^{h_1h_2\,10}\right)^{\sigma\,t\,\tau\;-\frac{2}{3}\,0\,0}_{\sigma\,t\,\tau\;-\frac{2}{3}\,0\,0} = q^{\sigma} \tag{24}
$$

$$
\left(R_q^{h_1h_2\,10}\right)^{\sigma\,t\,\tau\;\frac{1}{3}\,\frac{1}{2}\,\frac{1}{2}}_{\sigma\,t\,\tau\;\frac{1}{3}\,\frac{1}{2}\,\frac{1}{2}} = q^{-\frac{1}{2}\sigma-\tau} \tag{25}
$$

$$
\left(R_q^{h_1h_2\,10}\right)^{\sigma\,t\,\tau\;\frac{1}{3}\,\frac{1}{2}\,-\frac{1}{2}}_{\sigma\,t\,\tau\;\frac{1}{3}\,\frac{1}{2}\,-\frac{1}{2}} = q^{-\frac{1}{2}\sigma+\tau} \tag{26}
$$

$$
\left(R_q^{h_1h_2\,10}\right)^{\sigma\,t\,\tau-1\;\;\frac{1}{3}\,\frac{1}{2}\,\frac{1}{2}}_{\sigma\,t\,\tau\;\;\frac{1}{3}\,\frac{1}{2}\,-\frac{1}{2}} = q^{-\frac{1}{2}\sigma+\frac{1}{2}}(q^{-1}-q)\{[t+\tau][t-\tau+1]\}^{\frac{1}{2}} \tag{27}
$$

$$
\left(R_q^{h_1h_2\,10}\right)^{\sigma-1\;t+\frac{1}{2}\;\tau-\frac{1}{2}\;\;\frac{1}{3}\,\frac{1}{2}\,\frac{1}{2}}_{\sigma\,t\,\tau\;\;-\frac{2}{3}\,0\,0}
$$

$$
= q^{\frac{1}{4}\sigma+\frac{1}{2}\tau+t+\frac{3}{2}}(q-q^{-1})
$$

$$
\times\{[t-\tau+1][\tfrac{1}{3}h_1+\tfrac{1}{3}h_2+\tfrac{1}{2}\sigma-t][-\tfrac{1}{3}h_1+\tfrac{2}{3}h_2-\tfrac{1}{2}\sigma+t+1]
$$

$$
\times[\tfrac{2}{3}h_1-\tfrac{1}{3}h_2-\tfrac{1}{2}\sigma+t+2]/[2t+2][2t+1]\}^{\frac{1}{2}} \tag{28}
$$

$$
\left(R_q^{h_1h_2\,10}\right)^{\sigma-1\;t-\frac{1}{2}\;\tau-\frac{1}{2}\;\;\frac{1}{3}\,\frac{1}{2}\,\frac{1}{2}}_{\sigma\,t\,\tau\;\;-\frac{2}{3}\,0\,0}
$$

$$
= q^{\frac{1}{4}\sigma+\frac{1}{2}\tau-t+\frac{1}{2}}(q^{-1}-q)
$$

$$
\times\{[t+\tau][\tfrac{1}{3}h_1-\tfrac{2}{3}h_2+\tfrac{1}{2}\sigma+t][\tfrac{1}{3}h_1+\tfrac{1}{3}h_2+\tfrac{1}{2}\sigma+t+1]
$$

$$
\times[\tfrac{2}{3}h_1-\tfrac{1}{3}h_2-\tfrac{1}{2}\sigma-t+1]/[2t][2t+1]\}^{\frac{1}{2}} \tag{29}
$$

$$
\begin{aligned}
&\left(R_q^{h_1 h_2 10}\right)^{\sigma-1\, t+\frac{1}{2}\, \tau+\frac{1}{2}\ \frac{1}{3}\, \frac{1}{2}\, -\frac{1}{2}}_{\sigma\, t\, \tau\quad -\frac{2}{3}\, 0\, 0} \\
&\quad = q^{\frac{1}{4}\sigma+\frac{1}{2}\tau+\frac{1}{2}}(q^{-1}-q) \\
&\qquad \times \{[t+\tau+1][\tfrac{1}{3}h_1+\tfrac{1}{3}h_2+\tfrac{1}{2}\sigma-t][-\tfrac{1}{3}h_1+\tfrac{2}{3}h_2-\tfrac{1}{2}\sigma+t+1] \\
&\qquad \times [\tfrac{2}{3}h_1-\tfrac{1}{3}h_2-\tfrac{1}{2}\sigma+t+2]/[2t+2][2t+1]\}^{\frac{1}{2}}
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
&\left(R_q^{h_1 h_2 10}\right)^{\sigma-1\, t-\frac{1}{2}\, \tau+\frac{1}{2}\ \frac{1}{3}\, \frac{1}{2}\, -\frac{1}{2}}_{\sigma\, t\, \tau\quad -\frac{2}{3}\, 0\, 0} \\
&\quad = q^{\frac{1}{4}\sigma+\frac{1}{2}\tau+\frac{1}{2}}(q^{-1}-q) \\
&\qquad \times \{[t-\tau][\tfrac{1}{3}h_1-\tfrac{2}{3}h_2+\tfrac{1}{2}\sigma+t][\tfrac{1}{3}h_1+\tfrac{1}{3}h_2+\tfrac{1}{2}\sigma+t+1] \\
&\qquad \times [\tfrac{2}{3}h_1-\tfrac{1}{3}h_2-\tfrac{1}{2}\sigma-t+1]/[2t][2t+1]\}^{\frac{1}{2}}
\end{aligned} \tag{31}
$$

The pentagonal equation gives nine recursion relations for the general R-matrices six of which are summarized below:

$$
\begin{aligned}
&\left(R_q^{\lambda' \eta}\right)^{\sigma-i\, t-l\, \tau-j\quad \rho\, s\, \nu}_{\sigma\, t\, \tau\quad \rho-i\, s-m\, \nu-j}\ {}_q\langle \lambda\, \sigma+\tfrac{2}{3}\, t\, \tau; \epsilon\, -\tfrac{2}{3}\, 0\, 0|\lambda' \sigma\, t\, \tau\rangle \\
&\quad = \left(R_q^{\lambda \eta}\right)^{\sigma+\frac{2}{3}-i\, t-l\, \tau-j\quad \rho\, s\, \nu}_{\sigma\, t\, \tau\quad \rho-i\, s-m\, \nu-j} \left(R_q^{\epsilon \eta}\right)^{-\frac{2}{3}00\ \rho s\nu}_{-\frac{2}{3}00\ \rho s\nu} \\
&\qquad \times\ {}_q\langle \lambda\, \sigma+\tfrac{2}{3}-i\, t-l\, \tau-j; \epsilon\, -\tfrac{2}{3}\, 0\, 0|\lambda'\, \sigma-i\, t-l\, \tau-j\rangle
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&\left(R_q^{\lambda' \eta}\right)^{\sigma-i\, t-l\, \tau-j\quad \rho\, s\, \nu}_{\sigma\, t+\frac{1}{2}\, \tau\quad \rho-i\, s-m\, \nu-j} \\
&\qquad \times\ {}_q\langle \lambda\, \sigma-\tfrac{1}{3}\, t\, \tau-\tfrac{1}{2}; \epsilon\, \tfrac{1}{3}\, \tfrac{1}{2}\, \tfrac{1}{2}|\lambda'\, \sigma\, t+\tfrac{1}{2}\, \tau\rangle \\
&\qquad + \left(R_q^{\lambda' \eta}\right)^{\sigma-i\, t-l\, \tau-j\quad \rho\, s\, \nu}_{\sigma\, t-\frac{1}{2}\, \tau\quad \rho-i\, s-m\, \nu-j} \\
&\qquad \times\ {}_q\langle \lambda\, \sigma-\tfrac{1}{3}\, t\, \tau-\tfrac{1}{2}; \epsilon\, \tfrac{1}{3}\, \tfrac{1}{2}\tfrac{1}{2}|\lambda'\, \sigma\, t-\tfrac{1}{2}\, \tau\rangle \\
&\quad = \left(R_q^{\lambda \eta}\right)^{\sigma+\frac{2}{3}-i\, t-l\, \tau-j\ \rho-1\, s+\frac{1}{2}\, \nu-\frac{1}{2}}_{\sigma-\frac{1}{3}\, t\, \tau-\frac{1}{2}\quad \rho-i\, s-m\, \nu-j} \left(R_q^{\epsilon \eta}\right)^{-\frac{2}{3}00\quad \rho s\nu}_{\frac{1}{3}\frac{1}{2}\frac{1}{2}\ \rho-1\, s+\frac{1}{2}\, \nu-\frac{1}{2}} \\
&\qquad \times\ {}_q\langle \lambda\, \sigma+\tfrac{2}{3}-i\, t-l\, \tau-j; \epsilon\, 0\, 0|\lambda'\, \sigma-i\, t-l\, \tau-j\rangle \\
&\qquad + \left(R_q^{\lambda \eta}\right)^{\sigma+\frac{2}{3}-i\, t-l\, \tau-j\ \rho-1\, s-\frac{1}{2}\, \nu-\frac{1}{2}}_{\sigma-\frac{1}{3}\, t\, \tau-\frac{1}{2}\quad \rho-i\, s-m\, \nu-j} \left(R_q^{\epsilon \eta}\right)^{-\frac{2}{3}00\quad \rho s\nu}_{\frac{1}{3}\frac{1}{2}\frac{1}{2}\ \rho-1\, s-\frac{1}{2}\, \nu-\frac{1}{2}} \\
&\qquad \times\ {}_q\langle \lambda\, \sigma+\tfrac{2}{3}-i\, t-l\, \tau-j; \epsilon\, 0\, 0|\lambda'\, \sigma-i\, t-l\, \tau-j\rangle \\
&\qquad + \left(R_q^{\lambda \eta}\right)^{\sigma-\frac{1}{3}-i\, t+\frac{1}{2}-l\, \tau-\frac{1}{2}-j\quad \rho\, s\, \nu}_{\sigma-\frac{1}{3}\, t\, \tau-\frac{1}{2}\quad \rho-i\, s-m\, \nu-j} \left(R_q^{\epsilon \eta}\right)^{\frac{1}{3}\frac{1}{2}\frac{1}{2}\ \rho s\nu}_{\frac{1}{3}\frac{1}{2}\frac{1}{2}\ \rho s\nu} \\
&\qquad \times\ {}_q\langle \lambda\, \sigma-\tfrac{1}{3}-i\, t+\tfrac{1}{2}-l\, \tau-\tfrac{1}{2}-j; \epsilon\, \tfrac{1}{3}\, \tfrac{1}{2}\, \tfrac{1}{2}|\lambda'\, \sigma-i\, t-l\, \tau-j\rangle \\
&\qquad + \left(R_q^{\lambda \eta}\right)^{\sigma-\frac{1}{3}-i\, t-\frac{1}{2}-l\, \tau-\frac{1}{2}-j\quad \rho\, s\, \nu}_{\sigma-\frac{1}{3}\, t\, \tau-\frac{1}{2}\quad \rho-i\, s-m\, \nu-j} \left(R_q^{\epsilon \eta}\right)^{\frac{1}{3}\frac{1}{2}\frac{1}{2}\ \rho s\nu}_{\frac{1}{3}\frac{1}{2}\frac{1}{2}\ \rho s\nu} \times
\end{aligned}
$$

$$
\begin{aligned}
&\times\ {}_q\langle\lambda\ \sigma-\tfrac{1}{3}-i\ t-\tfrac{1}{2}-l\ \tau-\tfrac{1}{2}-j;\epsilon\ \tfrac{1}{3}\ \tfrac{1}{2}\ \tfrac{1}{2}|\lambda'\ \sigma-i\ t-l\ \tau-j\rangle\\
&+\left(R_q^{\lambda\,\eta}\right)^{\sigma-\frac{1}{3}-i\ t+\frac{1}{2}-l\ \tau+\frac{1}{2}-j\qquad \rho\ s\ \nu-1}_{\sigma-\frac{1}{3}\ t\ \tau-\frac{1}{2}\qquad \rho-i\ s-m\ \nu-j}\left(R_q^{\epsilon\,\eta}\right)^{\frac{1}{3}\,\frac{1}{2}\,-\frac{1}{2}\quad \rho s\nu}_{\frac{1}{3}\,\frac{1}{2}\,\frac{1}{2}\quad \rho s\nu-1}\\
&\times\ {}_q\langle\lambda\ \sigma-\tfrac{1}{3}-i\ t+\tfrac{1}{2}-l\ \tau+\tfrac{1}{2}-j;\epsilon\ \tfrac{1}{3}\ \tfrac{1}{2}\ -\tfrac{1}{2}|\lambda'\ \sigma-i\ t-l\ \tau-j\rangle\\
&+\left(R_q^{\lambda\,\eta}\right)^{\sigma-\frac{1}{3}-i\ t-\frac{1}{2}-l\ \tau+\frac{1}{2}-j\qquad \rho\ s\ \nu-1}_{\sigma-\frac{1}{3}\ t\ \tau-\frac{1}{2}\qquad \rho-i\ s-m\ \nu-j}\left(R_q^{\epsilon\,\eta}\right)^{\frac{1}{3}\,\frac{1}{2}\,-\frac{1}{2}\quad \rho s\nu}_{\frac{1}{3}\,\frac{1}{2}\,\frac{1}{2}\quad \rho s\nu-1}\\
&\times\ {}_q\langle\lambda\ \sigma-\tfrac{1}{3}-i\ t-\tfrac{1}{2}-l\ \tau+\tfrac{1}{2}-j;\epsilon\ \tfrac{1}{3}\ \tfrac{1}{2}\ -\tfrac{1}{2}|\lambda'\ \sigma-i\ t-l\ \tau-j\rangle
\end{aligned}\tag{33}
$$

where we substitute $\lambda' = (h_1+1, h_2)$ to obtain equations (32*a*) and (33*a*), $\lambda' = (h_1, h_2+1)$ to obtain (32*b*) and (33*b*), and $\lambda' = (h_1-1, h_2-1)$ to obtain (32*c*) and (33*c*).

Recursion relation (33*a*) on iterating h_1 times gives an expression for

$$
\left(R_q^{h_1\,0\,g_1\,0}\right)^{\frac{1}{3}h_1-i\ \frac{1}{2}h_1-\frac{1}{2}i\ \frac{1}{2}h_1-j\qquad \frac{1}{3}g_1\ \frac{1}{2}g_1\ \nu}_{\frac{1}{3}h_1\ \frac{1}{2}h_1\ \frac{1}{2}h_1\qquad \frac{1}{3}g_1-i\ \frac{1}{2}g_1-\frac{1}{2}i\ \nu-j}
$$

as a sum over terms of the form

$$
\left(R_q^{0\,0\,g_1\,0}\right)^{-i+b\ -\frac{1}{2}i+\frac{1}{2}b\ -j+c+\frac{1}{2}b\ \ \frac{1}{3}g_1-b\ \frac{1}{2}g_1-\frac{1}{2}b\ \nu-c-\frac{1}{2}b}_{0\,0\,0\qquad\qquad \frac{1}{3}g_1-i\ \frac{1}{2}g_1-\frac{1}{2}i\ \nu-j}.
$$

The only such term which is non-zero is the trivial R-matrix for which $b = i$, and $c = j - \frac{1}{2}i$. This matrix can be substituted for, to obtain the left-hand side.

Equation (32*a*) is used to obtain R-matrices with general σ in terms of those with $\sigma = \frac{1}{3}h_1$.

Rearranging (33*a*), moving the third term on the right to the left-hand side, and iterating $s-\nu$ times gives

$$
\left(R_q^{h_1\,0\,g_1\,0}\right)^{\frac{1}{3}h_1-i\ \frac{1}{2}h_1-\frac{1}{2}i\ \tau-j\qquad \rho\ s\ \nu}_{\frac{1}{3}h_1\ \frac{1}{2}h_1\ \tau\qquad \rho-i\ s-\frac{1}{2}i\ \nu-j}
$$

as a sum over R-matrices of the form obtained in the previous step, namely with $\tau = t$. The expression for these can be substituted and one of the sums performed using the q-equivalent of the binomial coefficient sum rule.

Iterating (32*a*) $\frac{2}{3}h_1 - \frac{1}{3}h_2 - \frac{1}{2}\sigma - t$ times, (32*b*) $-\frac{1}{3}h_1 + \frac{2}{3}h_2 - \frac{1}{2}\sigma + t$ times and (32*c*) $\frac{1}{3}h_1 + \frac{1}{3}h_2 + \frac{1}{2}\tau - t$ times gives

$$
\left(R_q^{h_1\,h_2\,g_1\,0}\right)^{\sigma-i\ t-\frac{1}{2}i\ \tau-j\qquad \frac{1}{3}g_1\ \frac{1}{2}g_1\ \nu}_{\sigma\ t\ \tau\qquad \frac{1}{3}g_1-i\ \frac{1}{2}g_1-\frac{1}{2}i\ \nu-j}
$$

with general σ, t and h_2 in terms of R-matrices with $h_2 = 0$, $\sigma = \frac{1}{3}h_1$ and $t = \frac{1}{2}h_1$.

Finally, using crossing symmetry (5) and iterating in a similar manner to the previous step, we have the R-matrices where the only restrictions are $l = m = \frac{1}{2}i$ and $j \geqslant 0$, namely

$$
\begin{aligned}
&\left(R_q^{h_1\,h_2\,g_1\,g_2}\right)^{\sigma-i\ t-\frac{1}{2}i\ \tau-j\qquad \rho\ s\ \nu}_{\sigma\ t\ \tau\qquad \rho-i\ s-\frac{1}{2}i\ \nu-j}\\
&\qquad = q^{-\frac{3}{2}\sigma\rho+\frac{3}{4}i\sigma+\frac{3}{4}i\rho-\frac{1}{2}ij+\frac{1}{4}i+\frac{1}{2}j-\frac{1}{8}i^2-\frac{1}{2}j^2+2ts}\times
\end{aligned}
$$

$$
\begin{aligned}
&\times q^{-2t\nu-2s\tau-\frac{1}{2}si+\frac{1}{2}\nu i-it+i\tau+j\tau+js+s-\nu} \\
&\times (q^{-1}-q)^{j+\frac{1}{2}i}\sum_z \frac{q^{-2tz+2\tau z+\frac{1}{2}iz-jz-z}(-)^{z-s+\nu}[j+\frac{1}{2}i+z]!}{[z]![s-\nu-z]![-s+\nu+i+z]![j+\frac{1}{2}i]!} \\
&\times \left\{\frac{[2t-i]![2t-i+1]![t+\tau]![t-\frac{1}{2}i-\tau+j]!}{[2t]![2t+1]![t-\tau]![t-\frac{1}{2}i+\tau-j]!}\right. \\
&\times \frac{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t]!}{[\frac{1}{3}h_1-\frac{2}{3}h_2+\frac{1}{2}\sigma+t-i]!} \\
&\times \frac{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t+1]![\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t+i]!}{[\frac{1}{3}h_1+\frac{1}{3}h_2+\frac{1}{2}\sigma+t-i+1]![\frac{2}{3}h_1-\frac{1}{3}h_2-\frac{1}{2}\sigma-t]!} \\
&\times \frac{[2s-i]![2s-i+1]![s+\nu]![s-\nu]!}{[2s]![2s+1]![s-\frac{1}{2}i+\nu-j]![s-\frac{1}{2}i-\nu+j]!} \\
&\times \frac{[\frac{2}{3}g_1-\frac{1}{3}g_2-\frac{1}{2}\rho-s+i]!}{[\frac{2}{3}g_1-\frac{1}{3}g_2-\frac{1}{2}\rho-s]!} \\
&\times \left.\frac{[\frac{1}{3}g_1-\frac{2}{3}g_2+\frac{1}{2}\rho+s]![\frac{1}{3}g_1+\frac{1}{3}g_2+\frac{1}{2}\rho+s+1]!}{[\frac{1}{3}g_1-\frac{2}{3}g_2+\frac{1}{2}\rho+s-i]![\frac{1}{3}g_1+\frac{1}{3}g_2+\frac{1}{2}\rho+s-i+1]!}\right\}^{\frac{1}{2}}. \qquad (34)
\end{aligned}
$$

6. Conclusions

The pentagonal relation provides recursion equations for calculating R-matrices. Unlike previous methods which require the complete set of vector coupling coefficients to calculate R-matrices, in the present method only the primitive coefficients are required for any R-matrices.

The resulting $\mathrm{su}(2)_q$ calculation involves only the straightforward solution of a recursion relation and thus is more systematic than Nomura (1989).

A complete class of R-matrices has been found by the recursive method for $\mathrm{su}(3)_q$. As a first step, the algebraic form of the primitive vector coupling coefficients were obtained by the method in Lienert and Butler (1992). Our results agree with those matrix elements and primitive vector coupling coefficients obtained by Ma (1990a, b), namely for the cases of $(h_1, h_2) = (g_1, g_2) = (1,0)$, $(2,0)$ and $(2,1)$. Calculation of the complete form of $\mathrm{su}(3)_q$ R-matrices is algebraically involved, but can be obtained in the same manner as the results given here.

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